Minilessons for Early Multiplication and Division

A Yearlong Resource

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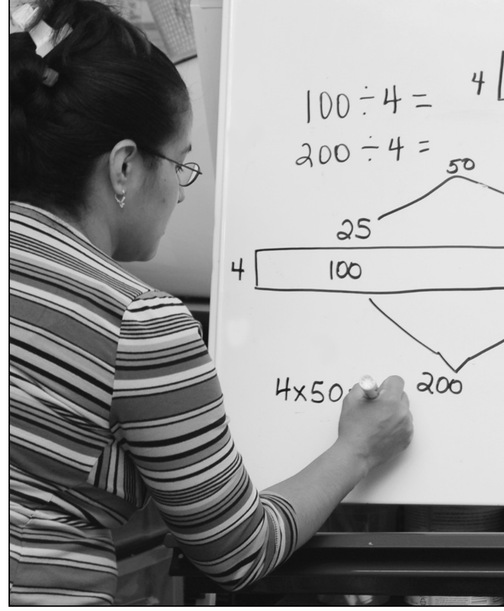


Unlike many of the other units in the *Contexts for Learning Mathematics* series, which consist of two-week sequences of investigations and related minilessons, this unit is meant to be used as a resource of 75 minilessons that you can choose from throughout the year. In contrast to investigations, which constitute the heart of the math workshop, the minilesson is more guided and more explicit, designed to be used at the start of math workshop and to last for ten to fifteen minutes. Each day, no matter what other unit or materials you are using, you might choose a minilesson from this resource to help your students develop efficient computation. You can also use the minilessons with small groups of students as you differentiate instruction.

This guide is structured progressively, moving from the use of pictures, where the contexts can be especially helpful, to the use of the number line, the ratio table, and finally the array. Although you may not use every minilesson in this resource, you will want to work through it with a developmental progression in mind.

The last section of the guide focuses on the relationship of division to multiplication. It is placed last because teachers often prefer to develop a deep understanding of multiplication and to automatize the basic facts before they work on division. It is possible, however, to work on the two operations simultaneously, and then this last section can be used by integrating the minilessons found in it into the sections with the corresponding models.

Some of the minilessons in this unit make use of carefully designed pictures that support the development of important strategies for multiplication and division by building in potentially realizable strategies or constraints. For example, the picture of the baker figuring out how many muffins he has left may support the development of the distributive property *(See Figure 1)*. The number of muffins in the first two trays is equal to the number in the third tray. The picture also may support the use of ten-times to determine nine-times, since the muffins in the third tray (9×4) are nested within a tray that holds ten rows of four. Although the picture is designed with these strategies in mind, however, that does not ensure that students will use these strategies. That is why we call them potentially realizable.

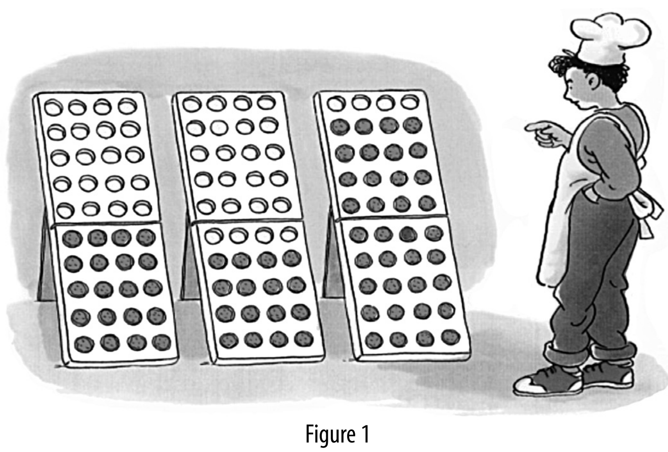


Figure 1

The picture of four tiled patios is designed with a constraint *(See Figure 2)*. In the first patio, every tile is visible and a counting strategy can therefore be used to determine the total. However, the furniture obscures some of the tiles in the other patios, providing a constraint to a counting strategy and supporting the use of the distributive property.

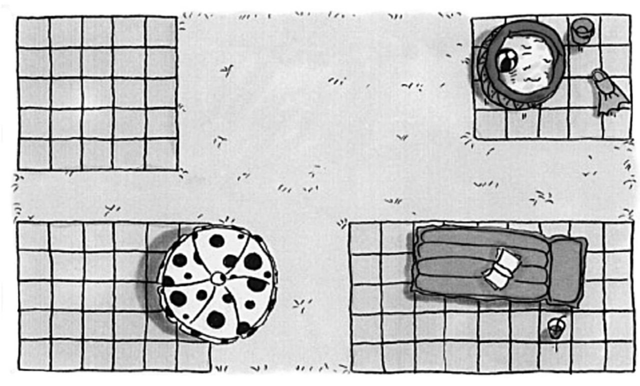


FIGURE 2

Other minilessons in this resource unit are crafted as "strings" of computation problems designed to encourage students to look to the numbers first, before they decide on a computation strategy. These minilessons will support your students in automatizing the basic facts while simultaneously developing numeracy. Each string is a tightly structured series of problems likely to generate discussion of certain strategies or big ideas underlying an understanding of multiplication and division.

Using Models during Minilessons

As you do minilessons from this resource unit, you will want to use models to depict students' strategies. Number lines, ratio tables, and arrays are most helpful for early multiplication and division. Initially, students think of multiplication as repeated addition and they either count by ones, add, or skip-count. These strategies map out well onto a number line since the repeated groups can be represented as jumps of equal length, and as you explore the relationships among the problems in the strings, equivalent expressions can be placed above and below the line. The pictures in the first section of this guide can be used to encourage students to develop other strategies, such as using partial products, and you can circle the groups right on the overhead transparencies of the pictures. The ratio table supports multiplicative (proportional) reasoning, particularly when used with contexts. For example, a t-chart representing days and weeks is a useful context for exploring multiplication by seven. If students are challenged by the multiplicative reasoning on the ratio table, each group (week) can be represented until students are able to jump flexibly around the ratio table—for example, going from 4 weeks and 28 days to 8 weeks and 56 days in one step, by doubling. The array is the most difficult model for students to understand because rows and columns must be coordinated (Battista et al. 1998). The square units of area are in both rows and columns simultaneously and this is often difficult for students to understand.

It is assumed that some work with these models has already been done with realistic situations and rich investigations. In the *Contexts for Learning Mathematics* series, *Measuring for the Art Show* and *Ages and Timelines* develop the open number line model, *Muffles' Truffles* develops the array model, and *The Big Dinner* develops the ratio table model. If your students do not have a well-developed understanding of these models, you may find it beneficial to use these units first before you do the minilessons that employ the models. Representing computation strategies with mathematical models provides students with images for discussion and supports the capacity to use a variety of strategies for computational fluency, but only if the models are understood. Once a model has been developed as a representation of a realistic situation, you can use it to record the computation strategies that students use.

The Mathematical Landscape: Developing Numeracy

Once students have developed an understanding of the operation of multiplication, emphasis has traditionally been placed on memorizing the basic facts through repetitive drill and practice, using worksheets and flash cards. Is it necessary for students to memorize facts? Certainly. In order to multiply numbers with double or triple digits quickly, students need to know the basic facts. But the debate in our schools often centers on understanding versus memorization, as if the approaches are dichotomies: students either count on their fingers or memorize isolated facts. Students need to understand what it means to multiply and divide before facts can become automatic, but understanding does not necessarily lead to this automaticity. In other words, understanding is necessary but not sufficient. Students often develop a good understanding of what it means to multiply two numbers, and they demonstrate this understanding by using their fingers, cubes, or drawings to depict repeated addition. Even with this understanding, however, they count several times—first each group, then the total. Even when students construct more efficient strategies like skip-counting or doubling, they may still rely on counting with their fingers to keep track of the groups.

Although these strategies are useful beginning points, students cannot be left with only these limited methods for solving multiplication and division problems. But is the answer the memorization of isolated facts? How many facts are there? And how do we help students understand the relationships between facts, like $9 \times 7 = (10 \times 7) - 7$?

Many students who struggle to commit basic facts to memory believe that there are "hundreds" to be memorized because they have little or no understanding of the relationships among them. Students who commit the facts to memory easily are able to do so because they have constructed relationships among them and use these relationships as shortcuts. Here are some important strategies to develop:

- Doubling: $6 \times 6 = 2 \times 3 \times 6$
- Halving and doubling: $4 \times 3 = 2 \times 6$
- Using the distributive property:
  $7 \times 8 = (5 \times 8) + (2 \times 8)$, or
  $7 \times 8 = (8 \times 8) - (1 \times 8)$
- Using the distributive property with tens:
  $9 \times 8 = (10 \times 8) - 8$
- Using the commutative property:
  $5 \times 8 = 8 \times 5$

Memorizing facts with flash cards or through drill and practice on worksheets will not develop these relationships. When these strategies are understood and used, there are fewer facts to memorize; for example, the commutative property means that nearly half the facts are repeats. The result of 1 multiplied by another number is the other number, so the facts that have 1 as a factor do not have to be memorized either. And squared numbers are often easy for students to remember. If you add partial products and doubling and halving strategies, there are very few facts left to memorize.

Memorization or Automaticity?

Memorization of basic facts usually refers to committing the results of operations to memory so that thinking through a computation is unnecessary. Isolated multiplications and divisions are practiced one after another; the emphasis is on recalling the answers. Teaching facts for automaticity, in contrast, relies on thinking. Answers to facts must be automatic, produced in only a few seconds; counting each time to obtain an answer is not acceptable. But thinking about the relationships among the facts is critical. A student who thinks of 9×6 as $(10 \times 6) - 6$ produces the answer of 54 quickly, but thinking rather than memorization is the focus (although over time these facts are remembered). The issue here is not whether facts should eventually be memorized, but how this memorization should be achieved: by rote drill and practice or by focusing on relationships.

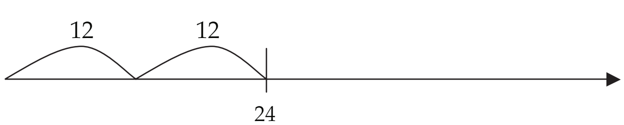
By making arrays on graph paper and overlaying these arrays one on top of another, students can explore relationships and write strategies to help themselves learn the facts that are difficult to remember. Pictures with constraints and mental math strings can also be used to develop understanding of these relationships, an understanding that leads to automaticity of the basic facts. Relationships among basic facts exist because of the properties of multiplication (commutative, associative, distributive, and identity), so this approach gets right to the heart of mathematics.

Using Minilessons to Develop Number Sense: An Example

Minilessons are usually done with the whole class together in a meeting area. Young children often sit on a rug; for older students, benches or chairs can be placed in a U-shape. Clustering students together like this, near a chalkboard or whiteboard, is helpful because you will want to provide an opportunity for pair talk at times, and you will need space to represent the strategies that will become the focus of discussion. The problems are written one at a time and learners are asked to determine an answer. Although the emphasis is on the development of mental arithmetic strategies, this does not mean learners have to solve the problems in their heads—but it is important for them to do the problem with their heads! In other words, encourage students to examine the numbers in the problem and let those numbers guide them in finding clever, efficient ways to reach a solution. The relationships among the problems in each minilesson will support them in doing this. By developing a repertoire of strategies, an understanding of the big ideas underlying why they work, and a variety of ways to model the relations, students are creating powerful toolboxes for flexible, efficient computation as well as automatizing the basic facts. Enter a classroom with us and see how this is done.

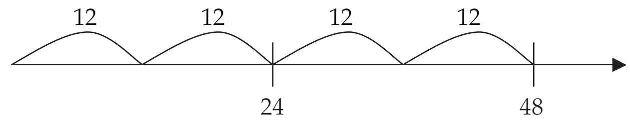
Each day at the start of math workshop, Trish Lent, a third-grade teacher in New York City, does a short minilesson on computation strategies. She usually chooses a string of six to eight related problems (like the ones provided in this resource unit) and asks her students to solve them, one at a time, and share their strategies with each other. She allows her students to construct their own strategies by decomposing numbers in ways that make sense to them. Posted on a strategy wall nearby are signs that the students have made throughout the year as they developed a repertoire of strategies for multiplication and division. The signs read, "Put helpful smaller pieces together,""Use doubles," and "When dividing, use multiplication." On the chalkboard today as we enter the classroom are Trish's first four problems: 2 × 3, 4 × 3, 8 × 3, and 8 × 6. The students have been discussing doubling and now Trish writes the fifth problem, 4 × 12.

"I know that 2 times 12 is 24 and so I was trying to double that to get 4 times. But that's hard." David, a student in Trish's class, is explaining how he is trying to solve the problem. Trish draws a number line to represent what he has said so far:



"Can you add 20 to 24?" She helps him break down the addition into friendlier partial sums.

"Yes, that's 34, 44…oh, so it's 48," David announces proudly, and Trish finishes the representation on the number line:



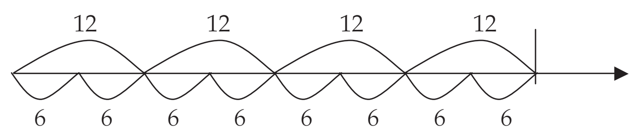
"How many people did this problem using David's method?" Trish asks. Most hands go up.

"It's the same answer as the other problem," another student, Linda, offers.

"It is, isn't it? Let me write that." Trish writes 4 × 12 = 8 × 6. "Could we find the 8 x 6 on this number line? How should I draw it?" Trish challenges the class to use the number line as a tool.

"I think every 12 has 2 sixes," Linda offers tentatively.

Trish draws in Linda's suggestion under the line.



"And David said that 2 twenty-fours equaled 4 twelves." Trish writes 2 × 24 = 4 × 12 = 8 × 6. "So what's happening here? Turn to the person next to you and talk about why you think these problems all have the same answer." Trish waits until she sees that most of the students are ready (indicated by thumbs-up) and then continues the discussion. "Rebecca?"

"It's like one number is doubling…the other is halving," says Rebecca. "It's like two for one. If you do two together, then you only have half as many numbers."

"That's pretty neat, isn't it? I wonder when it would be helpful to use this strategy. Here are two more problems. How can we make these friendly?" Trish writes 6 × 4, thinking that some students might use the equivalent expression of 3 × 8, and then she writes 9 × 7—a problem where doubling and halving will not help since they produce a fraction. Although Trish wants to develop a repertoire of strategies, she also wants students to think about differing situations for which certain strategies are (and are not) helpful.

She wants to encourage her young mathematicians to look to the numbers first before deciding on a strategy—this is the hallmark of numeracy.

A Few Words of Caution

As you work with the minilessons in this resource book, it is very important to remember two things. First, honor students' strategies. Accept alternative solutions and explore why they work. Use the models to represent students' strategies and facilitate discussion and reflection on the strategies shared. Sample classroom episodes (titled "Inside One Classroom") are interspersed throughout this resource guide to help you anticipate what learners might say and do and to provide you with images of teachers and students at work. The intent is not to get all learners to use the same strategy at the end of the string. That would simply be discovery learning. The strings are crafted to support development, to encourage students to look to the numbers, and to use a variety of strategies helpful for working with those numbers.

Secondly, do not use the string as a recipe that cannot be varied. You will need to be flexible. The strings are designed to encourage discussion and reflection on various strategies important for numeracy. Although the strings have been carefully crafted to support the development of these strategies, they are not foolproof: if the numbers in the string are not sufficient to produce the results intended, you will need to insert additional problems, depending on your students' responses, to finish the job. For this reason, most of the strings are accompanied by a Behind the Numbers section describing the string's purpose and how the numbers were chosen. Being aware of the purpose of each string will guide you in determining what further supports to add. These sections should also be helpful in developing your ability to craft your own strings. Strings are fun both to do and to craft.

References and Resources

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